ANALOGUES OF THE CENTRAL POINT THEOREM FOR FAMILIES WITH d-INTERSECTION PROPERTY IN $\mathbb{R}^d$

R.N. KARASEV

ABSTRACT. In this paper we consider families of compact convex sets in $\mathbb{R}^d$ such that any subfamily of size at most d has a nonempty intersection. We prove some analogues of the central point theorem and Tverberg's theorem for such families.

1. INTRODUCTION

Let us start with some definitions.

Definition 1.1. A family of sets $\mathcal{F}$ has *property* Π_k if for any nonempty $\mathcal{G} \subseteq \mathcal{F}$ such that $|\mathcal{G}| \leq k$ the intersection $\bigcap \mathcal{G}$ is not empty.

Helly's theorem [6] states that a finite family of convex sets (or any family of convex compact sets) with Π_{d+1} property in $\mathbb{R}^d$ has a common point. In the review [3] Helly's theorem and a lot of its generalizations are considered in detail.

In this paper we mostly consider the families with Π_d property in $\mathbb{R}^d$, the "almost" Helly property. The typical example of a family with Π_d property is any family of hyperplanes in general position. It can be easily seen that such a family need not have a common point, and even need not have a bounded piercing number (compare [9], where some bounds on piercing number are given for families of particular sets).

An important consequence of Helly's theorem is the central point theorem [4, 15, 16] for measures. Here we discuss its discrete version for finite point sets instead of measures.

Theorem (The discrete central point theorem)**.** *For a finite set $X \subset \mathbb{R}^d$ there exists a point $x \in \mathbb{R}^d$ such that any half-space $H \ni x$ contains at least*

$$r = \left\lceil \frac{|X|}{d+1} \right\rceil$$

points of X. Here $|X|$ denotes the cardinality of X.

In [10] a "dual" analogue of the central point theorem was established for the families of hyperplanes. Here it is proved for every family with Π_d property, and in a stronger form.

Theorem 1.1. *Let a finite family $\mathcal{F}$ of convex closed sets in $\mathbb{R}^d$ have property Π_d. Then there exists a point $x \in \mathbb{R}^d$ such that any unbounded continuous curve that passes through x intersects at least*

$$r = \left\lceil \frac{|\mathcal{F}|}{d+1} \right\rceil$$

sets in $\mathcal{F}$.

Similar to what is done in [10] it is natural to generalize this theorem in the spirit of Tverberg's theorem [17].

Definition 1.2. Consider a family $\mathcal{G}$ of $d+1$ convex compact sets in $\mathbb{R}^d$ with Π_d property. The family $\mathcal{G}$ either has a common point, or (by the Alexander duality [5], Theorem 3.44) the complement $\mathbb{R}^d \setminus \bigcup \mathcal{G}$ consists of two connected components: X and Y, where X is bounded and Y is unbounded. For a point $x \in X$ we say that $\mathcal{G}$ *surrounds* x.

Conjecture 1.1. *Let a finite family $\mathcal{F}$ of convex compact sets in $\mathbb{R}^d$ have property Π_d. Then there exists a point $x \in \mathbb{R}^d$ and*

$$r = \left\lceil \frac{|\mathcal{F}|}{d+1} \right\rceil$$

pairwise disjoint nonempty subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_r \subseteq \mathcal{F}$ such that the following condition holds for any $i = 1, \ldots, r$:
   1) either some member of $\mathcal{F}_i$ contains x;
   2) or the family $\mathcal{F}_i$ surrounds x.

It is easy to see that Conjecture 1.1 would imply Theorem 1.1, because any unbounded continuous curve through x must intersect some element of every $\mathcal{F}_i$. This conjecture remains open, but in order to deduce Theorem 1.1, it is sufficient (see Section 5) to prove Conjecture 1.2 for large enough set of values r, as it is done in the following theorem.

Theorem 1.2. *Conjecture 1.1 holds when r is a prime power.*

It is also possible to give a generalization in the spirit of Tverberg's transversal conjecture [18], see [2, 8, 10, 19, 21] for proofs of some particular cases of Tverberg's transversal conjecture and similar results.

Definition 1.3. Consider a family $\mathcal{G}$ of $d - m + 1$ convex compact sets in $\mathbb{R}^d$ with Π_{d-m} property and an m-flat L. We say that $\mathcal{G}$ *surrounds* L if $\pi(\mathcal{G})$ surrounds the point $\pi(L)$, where π is the projection along L.

Conjecture 1.2. *Suppose that each of $m+1$ families $\mathcal{F}_i$ $(i = 0, \ldots, m)$ of convex compact sets in $\mathbb{R}^d$ have property Π_{d-m}. Then there exists an m-flat L and, for every $i = 0, \ldots, m$*

$$r_i = \left\lceil \frac{|\mathcal{F}_i|}{d - m + 1} \right\rceil$$

pairwise disjoint nonempty subfamilies $\mathcal{F}_{i1}, \ldots, \mathcal{F}_{ir_i} \subseteq \mathcal{F}_i$ such that for any pair $i = 0, \ldots, m$, $j = 1, \ldots, r_i$ the following condition holds:
   1) either some member of $\mathcal{F}_{ij}$ intersects L;

2) or the family $\mathcal{F}_{ij}$ surrounds L.

Theorem 1.3. *Conjecture 1.2 is true when r_i are powers of the same prime p, and*
   1) either $p = 2$;
   2) or $d - m$ is even;
   3) or $m = 0$.

The case $m = 0$ is inserted here to make a unified statement with Theorem 1.2. Actually, in this theorem the sets need not be convex, it is sufficient that all their projections to $d - m$-flats are convex.

The restriction that r_i are prime powers is essential in the proof of Theorem 1.3, since the action of a p-torus on the configuration space is required, see Section 4.

2. Facts from topology

We consider topological spaces with continuous (left) action of a finite group G and continuous maps between such spaces that commute with the action of G. We call them G-spaces and G-maps. We mostly consider groups $G = (Z_p)^k$ for prime p here (p-tori).

For basic facts about (equivariant) topology and vector bundles the reader is referred to the books [7, 11, 14]. The cohomology is taken with coefficients in Z_p (p is the same as in the definition of G), in notations we omit the coefficients. Let us start from some standard definitions.

Definition 2.1. Denote EG the classifying G-space, which can be thought of as an infinite join $EG = G * \cdots * G * \ldots$ with diagonal left G-action. Denote $BG = EG/G$. For any G-space X denote $X_G = (X \times EG)/G$, and put (*equivariant cohomology in the sense of Borel*) $H_G^*(X) = H^*(X_G)$. It is easy to verify that for a free G-space X the space X_G is homotopy equivalent to X/G.

Consider the algebra of G-equivariant cohomology of the point $A_G = H_G^*(\mathrm{pt}) = H^*(BG)$. For a group $G = (Z_p)^k$ the algebra $A_G = H_G^*(Z_p)$ has the following structure (see [7]). In the case p odd it has $2k$ multiplicative generators v_i, u_i with dimensions $\dim v_i = 1$ and $\dim u_i = 2$ and relations

$$v_i^2 = 0, \quad \beta v_i = u_i.$$

We denote $\beta(x)$ the Bockstein homomorphism.

In the case $p = 2$ the algebra A_G is the algebra of polynomials of k one-dimensional generators v_i.

We are going to find the equivariant cohomology of a G-space X using the following spectral sequence (see [7, 13]).

Theorem 2.1. *The natural fiber bundle $\pi_{X_G} : X_G \to BG$ with fiber X gives the spectral sequence with E_2-term*

$$E_2^{x,y} = H^x(BG, \mathcal{H}^y(X)),$$

that converges to the graded module, associated with the filtration of $H_G^*(X)$.

The system of coefficients $\mathcal{H}^y(X)$ is obtained from the cohomology $H^y(X)$ by the action of $G = \pi_1(BG)$. The differentials of this spectral sequence are homomorphisms of $H^*(BG)$-modules.

This theorem implies that if the space X is $n - 1$-connected, than the natural map $A_G^m \to H_G^m(X)$ is injective in dimensions $m \leq n$.

Any representation of G can be considered as a vector bundle over the point pt, and it has corresponding characteristic classes in $H_G^*(\text{pt})$. We need the following lemma, that follows from the results of [7], Chapter III §1.

Lemma 2.1. *Let $G = (Z_p)^k$, and let $I[G]$ be the subspace of the group algebra $\mathbb{R}[G]$, consisting of elements*

$$\sum_{g \in G} a_g g, \quad \sum_{g \in G} a_g = 0.$$

Then the Euler class $e(I[G]) \neq 0 \in A_G$ and is not a divisor of zero in A_G.

Note that in this lemma the fact that $G = (Z_p)^k$ is essential.

We also need the following fact on the Grassmann variety from [2, 8, 21]. Consider the canonical bundle over the Grassmann variety $\gamma \to G_d^{d-m}$. In the case $p = 2$ we consider the variety of non-oriented $d - m$-subspaces, and for odd p we consider the variety of oriented subspaces.

Lemma 2.2. *For the Euler class $e(\gamma)$ modulo p the following holds*

$$e(\gamma)^m \neq 0 \in H^{m(d-m)}(G_d^{d-m}, Z_p),$$

if either $p = 2$, or $d - m$ is even, or $m = 0$. In the latter case we put $e(\gamma)^0 = 1 \in H^0(G_d^{d-m}, Z_p)$ by definition.

3. Topology of Tverberg's theorem

In Tverberg's theorem and its topological generalizations (see [1, 20] for example) it is important to consider the configuration space of r-tuples of points $x_1, \ldots, x_r \in \Delta^N$ with pairwise disjoint supports. Here Δ^N is a simplex of dimension N. Let us make some definitions, following the book [12].

Definition 3.1. Let K be a simplicial complex. Denote K_Δ^r the subset of the r-fold product K^r, consisting of the r-tuples $(x_1, \ldots, x_r)$ such that every pair x_i, x_j $(i \neq j)$ has disjoint supports in K. We call K_Δ^r the *r-fold deleted product of K*.

Definition 3.2. Let K be a simplicial complex. Denote K_Δ^{*r} the subset of the r-fold join K^{*r}, consisting of convex combinations $w_1 x_1 \oplus \cdots \oplus w_r x_r$ such that every pair x_i, x_j $(i \neq j)$ with weights $w_i, w_j > 0$ has disjoint supports in K. We call K_Δ^{*r} the *r-fold deleted join of K*.

Note that the deleted join is a simplicial complex again, while the deleted product has no natural simplicial complex structure, although it has some cellular complex structure.

The r-fold deleted product of the simplex $\Delta^{(r-1)(d+1)}$ is the natural configuration space in Tverberg's theorem, but sometimes it is simpler to use the deleted join because of the following fact. Denote $[r]$ the set $\{1, \ldots, r\}$ with the discrete topology, the following lemma is well-known, see [12] for example.

Lemma 3.1. *The deleted join of the simplex* $(\Delta^N)^{*r}_\Delta = [r]^{*N+1}$ *is* $N - 1$*-connected.*

If r is a prime power $r = p^k$, then the group $G = (Z_p)^k$ can be somehow identified with $[r]$, so a G-action on K^r_Δ and K^{*r}_Δ by permuting $[r]$ arises. In this case Theorem 2.1 and the above lemma imply that the natural map $A^l_G \to H^l_G((\Delta^N)^{*r}_\Delta)$ is injective in dimensions $l \leq N$. We need the similar fact for deleted products.

Lemma 3.2. *Let* $r = p^k$, $G = (Z_p)^k$, *and let* K *be a simplicial complex. If the natural map* $A^l_G \to H^l_G(K^{*r}_\Delta)$ *is injective for* $l \leq N$, *then the similar map* $A^l_G \to H^l_G(K^r_\Delta)$ *is injective for* $l \leq N - r + 1$.

Proof. Define the map $\alpha : K^{*r} \to \mathbb{R}[G]$ as follows. Let α map a convex combination $w_1 x_1 \oplus w_r x_r \in K^{*r}$ to $(w_1, \ldots, w_r) \in \mathbb{R}^r$, the latter space is identified with $\mathbb{R}[G]$, if we identify the set $[r]$ with G. This map is G-equivariant.

Consider the natural orthogonal projection $\pi : \mathbb{R}[G] \to I[G]$ (the latter G-representation is defined in Lemma 2.1) and the natural inclusion $\iota : K^{*r}_\Delta \to K^{*r}$. The map $\beta = \pi \circ \alpha \circ \iota : K^{*r}_\Delta \to I[G]$ is G-equivariant, and it can be easily checked that

$$K^r_\Delta = \{y \in K^{*r}_\Delta : \beta(y) = 0\}.$$

Now assume the contrary: the image of some nonzero $\xi \in A^l_G$ is zero in $H^l_G(K^r_\Delta)$ and $l \leq N - r + 1$. We denote the classes in A_G and their natural images in the equivariant cohomology of G-spaces by the same letters if it does not lead to confusion. Denote $e(I[G]) = e \in A^{r-1}_G$ for brevity. The Euler class of a vector bundle is zero outside the zero set of a section of the bundle, so $e = 0 \in H^{r-1}_G(K^{*r}_\Delta \setminus K^r_\Delta)$, and by the standard property of the cohomology product

$$e\xi = 0 \in H^{l+r-1}_G((K^{*r}_\Delta \setminus K^r_\Delta) \cup K^r_\Delta) = H^{l+r-1}_G(K^{*r}_\Delta).$$

By Lemma 2.1 $e\xi \neq 0 \in A^{l+r-1}_G$, and we come to contradiction with the injectivity condition in the statement of this lemma. $\square$

4. Proof of Theorem 1.3

It is sufficient to prove Theorem 1.3, since Theorem 1.2 is its particular case. The reasoning is essentially the same as in [10], compare also [8].

For any m-flat L denote the unique $d - m$-subspace in $\mathbb{R}^d$, orthogonal to L, by $L^\perp$. It is easy to see that L is determined uniquely by $L^\perp$ and the point $L \cap L^\perp$. So the variety of all m-flats is the total space of the canonical bundle γ^{d-m}_d over the Grassmann variety G^{d-m}_d.

Now consider some $\alpha \in G^{d-m}_d$ and a point $b \in \alpha$, denote the orthogonal projection onto α by π_α. For any $X \in \bigcup^m_{i=0} \mathcal{F}_i$ denote $\phi(b, X)$ the closest to b point in $\pi_\alpha(X)$. This point depends continuously on the pair (α, b).

Fix some $i = 0, \ldots, m$ and denote a linear map $\psi_i : K_i = \Delta^{|\mathcal{F}|+1} \to \alpha$, determined so that it maps the vertices of the simplex to the points $\phi(b, X) - b$ for $X \in \mathcal{F}_i$, and is piecewise linear. Denote $\xi_i : (K_i)_\Delta^{r_i} \to \alpha^{r_i}$ the corresponding map of the deleted products. Let the group $G_i = (Z_p)^{k_i}$, where $r_i = p^{k_i}$ act on the deleted product $L_i = (K_i)_\Delta^{r_i}$ and on α^{r_i} by permutations, we denote $\alpha^{r_i} = \alpha[G_i]$ to indicate this action, the map ξ_i is G_i-equivariant.

In the sequel we denote $\gamma_d^{d-m} = \gamma$ for brevity. Summing up all the maps we obtain a map

$$\xi : L_0 \times \cdots \times L_m \to \alpha[G_0] \oplus \cdots \oplus \alpha[G_m].$$

The map ξ also depends on the pair (α, b) continuously, so actually it gives a section ξ of the vector bundle

$$U = \alpha[G_0] \oplus \cdots \oplus \alpha[G_m] \to \gamma \times L_0 \times \cdots \times L_m.$$

Here α can be treated as the pullback of the vector bundle $\gamma \to G_d^{d-m}$ by the map $\pi : \gamma \to G_d^{d-m}$, so α is a vector bundle over γ.

To prove the theorem we have to find such $\alpha \in G_d^{d-m}, b \in \alpha, (y_0, \ldots, y_m) \in L_0 \times \cdots \times L_m$ that $\xi(\alpha, b, y_0, \ldots, y_m) = 0$.

If we take the bundle of large enough balls $B(\gamma)$ in γ, the section ξ obviously has no zeros on $\partial B(\gamma) \times L_0 \times \cdots \times L_m$. To guarantee the zeros for the section ξ, we have to find the relative Euler class (see [10] for properties of the relative Euler class)

$$e(\xi) \in H_{G_0 \times \cdots \times G_m}^{(d-m)(r_0 + \cdots + r_m)}(B(\gamma) \times L_0 \times \cdots \times L_m, \partial B(\gamma) \times L_0 \times \cdots \times L_m).$$

Denote for brevity $G = G_0 \times \cdots \times G_m$.

Let us decompose the bundle U and its section ξ in the following way. Any $\alpha[G_i]$ can be split $\alpha[G_i] = \alpha \otimes \mathbb{R}[G] = \alpha \otimes \mathbb{R} \oplus \alpha \otimes I[G_i] = \alpha \oplus \alpha \otimes I[G_i]$. So the ξ splits into section η of the bundle $V = \alpha^{m+1}$ and ζ of the bundle $W = \alpha \otimes \bigoplus_{i=0}^m I[G_i]$, and $U = V \oplus W$.

The section η has no zeroes on $\partial B(\gamma) \times L_0 \times \cdots \times L_m$ and, in fact, for large enough balls in $B(\gamma)$ the homotopy $\eta_t = (1 - t)\eta + t(-b, \ldots, -b)$ connects it to the section $(-b, \ldots, -b)$ so that η_t has no zeroes on $\partial B(\gamma) \times L_0 \times \cdots \times L_m$ for all $t \in [0, 1]$. The section η_1 does not depend on $L_0 \times \cdots \times L_m$ and it can be easily seen that (see [10], the proof of Theorem 6)

$$e(\eta) = u(\gamma)e(\gamma)^m \times 1 \in H^{(d-m)(m+1)}(B(\gamma), \partial B(\gamma)) \times H_G^0(L_0 \times \cdots \times L_m) \subset$$
$$\subset H^{(d-m)(m+1)}(B(\gamma) \times L_0 \times \cdots \times L_m, \partial B(\gamma) \times L_0 \times \cdots \times L_m),$$

where $u(\gamma)$ is the Thom's class of γ, $e(\gamma)$ (the same as $e(\alpha)$) is its Euler class, and the last inclusion is the Künneth formula. Lemma 2.1 shows that $u(\gamma)e(\gamma)^m \neq 0$ (compare [10], the proof of Theorem 6).

Now we consider the class $e(\zeta) \in H_G^{(d-m)(r_0 + \cdots + r_m - m - 1)}(B(\gamma) \times L_0 \times \cdots \times L_m)$. Taking some fixed $b \in B(\gamma)$ and considering the inclusion

$$\iota_b : L_0 \times \cdots \times L_m = \{b\} \times L_0 \times \cdots \times L_m \to B(\gamma) \times L_0 \times \cdots \times L_m$$

and the induced bundle $\iota_b^*(W) = \bigoplus_{i=0}^m (I[G_i])^{d-m}$, we obtain

$$\iota_b^*(e(\zeta)) = e(I[G_0])^{d-m} \times e(I[G_1])^{d-m} \times \cdots \times e(I[G_m])^{d-m} \in H_G^*(L_0 \times \cdots \times L_m) =$$
$$= H_{G_0}^*(L_0) \times \cdots \times H_{G_m}^*(L_m),$$

the last equality being the Künneth formula. By Lemmas 2.1 and 3.2, for any $i = 0, \ldots, m$ the Euler class $e(I[G])^{d-m} \neq 0 \in H_{G_i}^{(d-m)(r_i-1)}(L_i)$ and, by the Künneth formula, $\iota_b^*(e(\zeta)) = a \neq 0$. From one more Künneth formula for the product $B(\gamma) \times L_0 \times \cdots \times L_m$ it follows that

$$e(\zeta) = 1 \times a + \sum_j b_j \times c_j,$$

where $b_j \in H^*(B(\gamma)), c_j \in H_G^*(L_0 \times \cdots \times L_m)$, and $\dim b_j > 0$ for all j. So

$$e(\xi) = u(\gamma)e(\gamma)^m \times a + \sum_j u(\gamma)e(\gamma)^m b_j \times c_j,$$

and $e(\xi) \neq 0$ by the Künneth formula.

Now we can consider some zero of ξ. Let us have some subspace and a point $\alpha \ni b$, and $(y_0, \ldots, y_m) \in L_0 \times \cdots \times L_m$ such that $\xi(\alpha, b, y_0, \ldots, y_m) = 0$. Every point $y_i \in L_i$ is actually an r_i-tuple of points $y_{i1}, \ldots, y_{ir_i} \in K_i = \Delta^{|\mathcal{F}_i|+1}$ with pairwise disjoint supports. We identify the vertices of K_i with $\mathcal{F}_i$ and write

$$y_{ij} = \sum w(i, j, X)X.$$

Denote $\mathcal{F}_{ij} = \{X \in \mathcal{F}_i : w(i, j, X) > 0\}$, each X is assigned to no more than one of $\mathcal{F}_{ij}$, because y_{ij} have pairwise disjoint supports. The condition $\xi = 0$ implies that for any $i = 0, \ldots, m, \ j = 1, \ldots, r_i$ the point b is a convex combination of its projections

$$b = \sum_{X \in \mathcal{F}_{ij}} w(i, j, X)\phi(b, X).$$

If b coincides with one of $\phi(b, X)$, then L (the m-flat, perpendicular to α and passing through b) intersects the corresponding X. If b lies in the interior of the convex hull of some $d - m + 1$ points of $\phi(b, X)$, we change $\mathcal{F}_{ij}$ so that it contains only those $d - m + 1$ corresponding sets X and note, that $\{\pi_\alpha(X)\}_{X \in \mathcal{F}_{ij}}$ surround X by Lemma 4.1 (see below), and therefore $\mathcal{F}_{ij}$ surrounds L.

If none of the above alternatives holds, then b lies in the relative interior of the convex hull of some $n < d - m + 1$ points $\phi(b, X_1), \ldots, \phi(b, X_n)$, $X_1, \ldots, X_n \in \mathcal{F}_{ij}$. Denote the half-spaces

$$H_X = \{x \in \mathbb{R}^d : (x, \phi(b, X) - b) \geq (\phi(b, x), \phi(b, X) - b)\}.$$

Note that $X \subseteq H_X$ (since ϕ is the projection) and the half-spaces $H_{X_1}, \ldots, H_{X_n}$ have empty intersection. So some $n < d - m + 1$ sets of $\mathcal{F}_{ij}$ have an empty intersection, that contradicts the Π_{d-m} property.

Now we only have to prove the lemma.

Lemma 4.1. *Let a family* $\mathcal{G} = \{G_1, \ldots, G_{d+1}\}$ *of convex compact sets in* $\mathbb{R}^d$ *have property* Π_d. *Let a point* $b \in \mathbb{R}^d$ *be such that* b *lies in the interior of the convex hull of* $g_1, \ldots, g_{d+1}$, *where* g_i *is the closest to* b *point in* G_i. *Then* $\mathcal{G}$ *surrounds* b.

Proof. Again, denote the half-spaces

$$H_i = \{x \in \mathbb{R}^d : (x, g_i - b) \geq (g_i, g_i - b)\}$$

and note that $G_i \subseteq H_i$. Clearly, $\bigcap_{i=1}^{d+1} H_i = \emptyset$.

For any $i = 1, \ldots, d+1$ the nonempty intersection $\bigcap_{j \neq i} G_j$ is contained in $\bigcap_{j \neq i} H_i$, take one point $x_i \in \bigcap_{j \neq i} G_j$. The simplex $\Delta = \mathrm{conv}_{i=1}^{d+1}\{x_i\}$ contains $\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} H_i \ni b$ (compare [9], Lemma 1), and every its facet $\partial_i \Delta = \mathrm{conv}_{j \neq i}\{x_i\}$ is contained in the corresponding G_i.

Thus $b \notin \bigcup_{i=1}^{d+1} G_i$ and is separated from infinity by $\bigcup_{i=1}^{d+1} G_i \supseteq \partial \Delta$, so $\mathcal{G}$ surrounds b by definition. $\qquad\square$

5. Proof of Theorem 1.1

In this theorem we can assume that $\mathcal{F}$ consists of compact sets. Indeed, for a large enough ball B the family $\{X \cap B\}_{X \in \mathcal{F}}$ consists of compact sets and has property Π_d.

As it was already noted, this theorem follows from Theorem 1.2 directly when r is a prime power. Consider some other r. Obviously, it is sufficient to prove the theorem in the case $N = |\mathcal{F}| = (d+1)(r-1) + 1$.

By the Dirichlet theorem on arithmetic progressions, we can find a positive integer k such that $R = k(r-1) + 1$ is a prime. Now take the family $\mathcal{F}'$ of size kN by simply repeating each set in $\mathcal{F}$ exactly k times. Note that

$$kN = k(d+1)(r-1) + k = (d+1)(R-1) + k \geq (d+1)(R-1) + 1,$$

so we can apply the case of the theorem, that is already proved, to $\mathcal{F}'$ to get some point x.

Every unbounded closed curve $C \ni x$ intersects at least $R = k(r-1) + 1$ sets of $\mathcal{F}'$. Each set of $\mathcal{F}$ is counted no more that k times, then we conclude that C intersects at least r sets of $\mathcal{F}$.

6. Acknowledgments

References

[1] I. Bárány, S.B. Shlosman, S. Szücz. On a topological generalization of a theorem of Tverberg. // J. London Math. Soc., II. Ser., 23, 1981, 158–164.

[2] V.L. Dol'nikov. Common transversals for families of sets in $\mathbb{R}^n$ and connections between theorems of Helly and Borsuk. (In Russian) // Doklady Akademii Nauk USSR, 297(4), 1987, 777-780.

[3] J. Eckhoff. Helly, Radon, and Carathéodory type theorems. // Handbook of Convex Geometry, ed. by P.M. Gruber and J.M. Willis, North-Holland, Amsterdam, 1993, 389–448.

[4] B. Grünbaum. Partitions of mass-distributions and of convex bodies by hyperplanes. // Pacific J. Math., 10, 1960, 1257–1261.

[5] A. Hatcher. Algebraic Topology. Cambridge University Press, 2002.

[6] E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. // Jber Deutsch. Math. Verein., 32, 1923, 175–176.

[7] Wu Yi Hsiang. Cohomology theory of topological transformation groups. Springer Verlag, 1975.

[8] R.N. Karasev. Tverberg's transversal conjecture and analogues of nonembeddability theorems for transversals. // Discrete and Computational Geometry, 38(3), 2007, 513–525.

[9] R.N. Karasev. Piercing families of convex sets with d-intersection property in $\mathbb{R}^d$. // Discrete and Computational Geometry, 39(4), 2008, 766–777.

[10] R.N. Karasev. Dual theorems on central points and their generalizations. // Sbornik: Mathematics, 199(10), 2008, 1459–1479.

[11] G. Luke, A.S. Mishchenko. Vector bundles and their applications. Springer Verlag, 1998.

[12] J. Matoušek. Using the Borsuk-Ulam theorem. Berlin-Heidelberg, Springer Verlag, 2003.

[13] J. McCleary. A user's guide to spectral sequences. Cambridge University Press, 2001.

[14] J. Milnor, J. Stasheff. Characteristic classes. Princeton University Press, 1974.

[15] B.H. Neumann. On an invariant of plane regions and mass distributions. // J. London Math. Soc., 20, 1945, 226–237.

[16] R. Rado. A theorem on general measure. // J. London Math. Soc., 21, 1946, 291–300.

[17] H. Tverberg. A generalization of Radon's theorem. // J. London Math. Soc., 41, 1966, 123–128.

[18] H. Tverberg, S. Vrećica. On generalizations of Radon's theorem and the ham sandwich theorem. // Europ. J. Combinatorics, 14, 1993, 259–264.

[19] S.T. Vrećica. Tverberg's conjecture. // Discrete and Computational Geometry, 29, 2003, 505–510.

[20] A.Yu. Volovikov. On a topological generalization of the Tverberg theorem. // Mathematical Notes, 59(3), 1996, 324–326.

[21] R.T. Živaljević. The Tverberg-Vrećica problem and the combinatorial geometry on vector bundles. // Israel J. Math., 111, 1999, 53–76.

E-mail address: `r_n_karasev@mail.ru`

ROMAN KARASEV, DEPT. OF MATHEMATICS, MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, INSTITUTSKIY PER. 9, DOLGOPRUDNY, RUSSIA 141700